# Calculus 140, section 2.1 Tangent Lines and Velocity
notes prepared by Tim Pilachowski

We begin with the concept of **limits** in mathematics. Non-technically, taking a limit is moving constantly toward something without ever getting there. The first few homework exercises ask you to guess at the values of various limits.

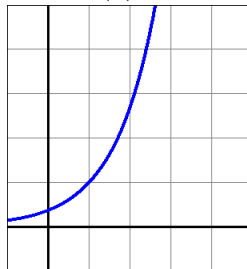The expression $\lim_{x \to a} f(x) = L$ is read "the limit of $f(x)$ as x approaches a equals L".

Examples A: For the following functions, guess the value of $\lim_{x \to 1} f(x)$.
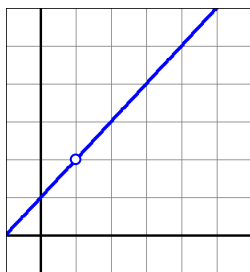
a) $f(x) = x - 1$
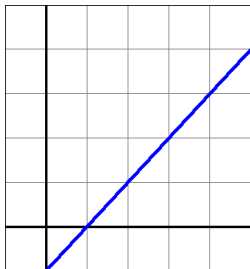
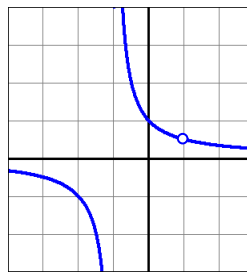b) $f(x) = e^{x-1}$

answers: 0, 1

In the case of rational functions, for which the denominator $\neq 0$, we can sometimes simplify first.

Examples B: For the following functions, guess the value of $\lim_{x \to 1} f(x)$.


a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

b) $f(x) = \dfrac{x - 1}{x^2 - 1}$

answers: 2, $\frac{1}{2}$

Toward the end of section 2.1, the text uses a table of decimal values to suggest that $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$. Although you won't have the tools to prove it until section 7.2, it is useful enough now to accept as fact and memorize.

In the rest of this section, we'll be concerned with using the concept of a limit to estimate ("guess") the slope of a line tangent to a graph.
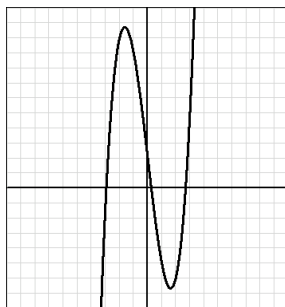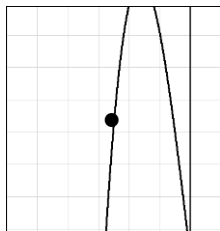
Consider the function $f(x)=x^3-8x+2$, pictured to the left. (To reproduce this on your graphing calculator set your window to [−10, 10] by [8, 12].)
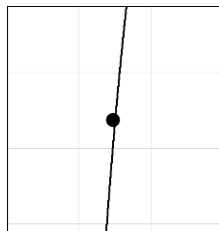
We'll estimate the slope of a tangent to the curve by traveling along the curve a piece at a time.

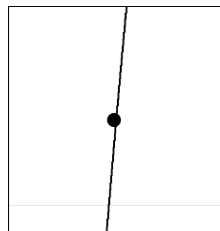Example C: The series of graphs below zoom in on the point $\left(-\dfrac{5}{2}, \dfrac{51}{8}\right)$. (You can duplicate this on your graphing calculator. Begin with the window in **Standard**. Use the **TRACE** key to get your cursor close to $x=-2.5$, press the **ZOOM** key, then choose **Zoom In**. Each time you press **ENTER** the calculator will show pictures similar to those below.) Dimensions of the graph window are given below each picture.
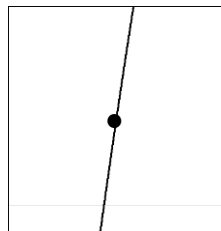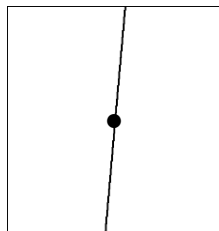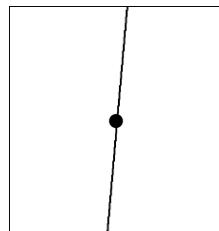

[− 6, 1] by	[− 4, − 1] by	[− 3, − 2] by	[− 2.8, − 2.2] by	[− 2.6, − 2.4] by	[− 2.55, − 2.45] by
[2.875, 9.875]	[4.875, 7.875]	[5.875, 6.875]	[6.075, 6.675]	[6.275, 6.475]	[6.325, 6.425]

We could continue the process, but the pictures would change very little. Notice what happens as we zoom in. It is clear in the original graph that at the point $\left(-\dfrac{5}{2}, \dfrac{51}{8}\right)$ the tangent has a steep positive slope. As we zoom in, the graph appears to have less curve and is more like a line.
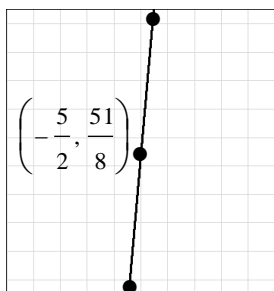

The final picture is reproduced to the left, with a grid marked in intervals of 0.01. We can roughly estimate the slope by approximating points and calculating using the formula for the slope of a line. The point at the top of the picture is approximately $(-2.495, 6.425)$. The point at the bottom of the picture is approximately $(-2.505, 6.325)$. We estimate the slope of the tangent to be $\dfrac{6.325-6.425}{-2.505-(-2.495)}=\dfrac{-0.1}{-0.01}=10$. This estimate is close, but is not exact—the slope of the tangent at this point is actually 10.75.

Example D: Given $f(x)=x^3-8x+2$, estimate the slope of the tangent at $x=-2$.
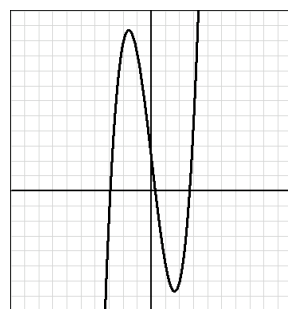
The visual method is mathematically insufficient, because even with the added precision of zooming in, we cannot be sure that our answer is *exact*, as opposed to being extremely close. Also, the method is not practical to reproduce for every point on a curve. However, if we turn to a sequence of secant lines (each of which connects a point $(a, f(a))$ to another point on the curve), the slope of each secant is $\dfrac{f(x)-f(a)}{x-a}$. As x approaches a, the sequence of secant lines approaches the tangent line at $x=a$, and the sequence of slopes approaches the slope of the tangent.

Example D revisited: Given $f(x) = x^3 - 8x + 2$, derive a formula for the slope of a tangent to the curve, then use this formula to determine the exact value of the slope of the tangent at $x = -2$.

answers: In mathematical notation: Given $f(x) = x^3 - 8x + 2$, $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{b - a} = 3x^2 - 8$; 4

And now for a topic you'll need for some homework exercises. (Consult the text's example along with the material below.)

If you were to drive from home to school, what would your initial velocity be?

What would your end velocity be?

What would your maximum velocity be?

Obviously, your velocity changes, ranging from a minimum of 0 mph to a maximum (assuming you obey speed limits) of 65 mph (assuming you travel on an interstate highway). How could you calculate your *average* velocity?

Given a function $f(t)$ that describes position, the average velocity between a time a and a later time t is given by

$$\text{average velocity} = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t) - f(a)}{t - a}.$$

Given a minimum velocity of 0 mph and a maximum velocity of 65 mph, at some moment (or moments) your velocity would be exactly 20 mph. At another instant, your velocity would be exactly 50 mph. How could you calculate your *instantaneous* velocity, i.e. your velocity at a specific instant or moment in time?

Given a function $f(t)$ that describes position, the velocity at a time t (instantaneous velocity) is given by

$$\text{velocity} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a}.$$